The Quantum Theory of the Electron.

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The new quantum mechanics, when applied to the problem of the structure of the atom with point-charge electrons, does not give results in agreement with experiment. The discrepancies consist of " duplexity " phenomena, the observed number of stationary states for an electron in an atom being twice the number given by the theory. To meet the difficulty, Goudsmit and Uhlenbeck have introduced the idea of an electron with a spin angular momentum of half a quantum and a magnetic moment of one Bohr magneton. This model for the electron has been fitted into the new mechanics by Pauli,* and Darwin,† working with an equivalent theory, has shown that it gives results in agreement with experiment for hydrogen-like spectra to the first order of accuracy.

The question remains as to why Nature should have chosen this particular model for the electron instead of being satisfied with the point-charge. One would like to find some incompleteness in the previous methods of applying quantum mechanics to the point-charge electron such that, when removed, the whole of the duplexity phenomena follow without arbitrary assumptions. In the present paper it is shown that this is the case, the incompleteness of the previous theories lying in their disagreement with relativity, or, alternatively, with the general transformation theory of quantum mechanics. It appears that the simplest Hamiltonian for a point-charge electron satisfying the requirements of both relativity and the general transformation theory leads to an explanation of all duplexity phenomena without further assumption. All the same there is a great deal of truth in the spinning electron model, at least as a first approximation. The most important failure of the model seems to be that the magnitude of the resultant orbital angular momentum of an electron moving in an orbit in a central field of force is not a constant, as the model leads one to expect.

* Pauli, ' Z. f. Physik,' vol. 43, p. 601 (1927).
† Darwin, ' Roy. Soc. Proc.,' A, vol. 116, p. 227 (1927).

§ 1. *Previous Relativity Treatments.*

The relativity Hamiltonian according to the classical theory for a point electron moving in an arbitrary electro-magnetic field with scalar potential $\mathbf{A_0}$ and vector potential $\mathbf{A}$ is

$$\mathbf{F} \equiv \left(\frac{\mathbf{W}}{c} + \frac{e}{c}\mathbf{A_0}\right)^2 + \left(\mathbf{p} + \frac{e}{c}\,\mathbf{A}\right)^2 + m^2c^2,$$

where $\mathbf{p}$ is the momentum vector. It has been suggested by Gordon[*] that the operator of the wave equation of the quantum theory should be obtained from this F by the same procedure as in non-relativity theory, namely, by putting

$$\mathbf{W} = ih\frac{\partial}{\partial t},$$

$$p_r = -ih\frac{\partial}{\partial x_r}, \qquad r = 1, 2, 3,$$

in it. This gives the wave equation

$$\mathbf{F}\psi \equiv \left[\left(ih\frac{\partial}{c\,\partial t} + \frac{e}{c}\,\mathbf{A_0}\right)^2 + \Sigma_r\left(-ih\frac{\partial}{\partial x_r} + \frac{e}{c}\,\mathbf{A}_r\right)^2 + m^2c^2\right]\psi = 0, \quad (1)$$

the wave function ψ being a function of x_1, x_2, x_3, t. This gives rise to two difficulties.

The first is in connection with the physical interpretation of ψ. Gordon, and also independently Klein,[†] from considerations of the conservation theorems, make the assumption that if ψ_m, ψ_n are two solutions

$$\rho_{mn} = -\frac{e}{2mc^2}\left\{ih\left(\psi_m\frac{\partial\overline{\psi}_n}{\partial t} - \overline{\psi}_n\frac{\partial\psi_m}{\partial t}\right) + 2e\mathbf{A_0}\psi_m\overline{\psi}_n\right\}$$

and

$$\mathbf{I}_{mn} = -\frac{e}{2m}\left\{-ih\left(\psi_m\operatorname{grad}\overline{\psi}_n - \overline{\psi}_n\operatorname{grad}\psi_m\right) + 2\frac{e}{c}\,\mathbf{A}_m\psi_m\overline{\psi}_n\right\}$$

are to be interpreted as the charge and current associated with the transition $m \to n$. This appears to be satisfactory so far as emission and absorption of radiation are concerned, but is not so general as the interpretation of the non-relativity quantum mechanics, which has been developed[‡] sufficiently to enable one to answer the question : What is the probability of any dynamical variable

[*] Gordon, ' Z. f. Physik,' vol. 40, p. 117 (1926).

[†] Klein, ' Z. f. Physik,' vol. 41, p. 407 (1927).

[‡] Jordan, ' Z. f. Physik,' vol. 40, p. 809 (1927) ; Dirac, ' Roy. Soc. Proc.,' A, vol. 113, p. 621 (1927).

at any specified time having a value lying between any specified limits, when the system is represented by a given wave function ψ_n?   The Gordon-Klein inter-pretation can answer such questions if they refer to the position of the electron (by the use of ρ_{nn}), but not if they refer to its momentum, or angular momentum or any other dynamical variable.   We should expect the interpretation of the relativity theory to be just as general as that of the non-relativity theory.

The general interpretation of non-relativity quantum mechanics is based on the transformation theory, and is made possible by the wave equation being of the form

$$(\mathrm{H} - \mathrm{W}) \, \psi = 0, \tag{2}$$

i.e., being linear in W or $\partial/\partial t$, so that the wave function at any time determines the wave function at any later time.   The wave equation of the relativity theory must also be linear in W if the general interpretation is to be possible.

The second difficulty in Gordon's interpretation arises from the fact that if one takes the conjugate imaginary of equation (1), one gets

$$\left[\left(-\frac{\mathrm{W}}{c} + \frac{e}{c} \, \mathrm{A}_0 \right)^2 + \left(-\mathbf{p} + \frac{e}{c} \, \mathbf{A} \right)^2 + m^2 c^2 \right] \psi = 0,$$

which is the same as one would get if one put $- e$ for e.   The wave equation (1) thus refers equally well to an electron with charge e as to one with charge $- e$.   If one considers for definiteness the limiting case of large quantum numbers one would find that some of the solutions of the wave equation are wave packets moving in the way a particle of charge $- e$ would move on the classical theory, while others are wave packets moving in the way a particle of charge e would move classically.   For this second class of solutions W has a negative value. One gets over the difficulty on the classical theory by arbitrarily excluding those solutions that have a negative W.   One cannot do this on the quantum theory, since in general a perturbation will cause transitions from states with W positive to states with W negative.   Such a transition would appear experi-mentally as the electron suddenly changing its charge from $- e$ to e, a phenomenon which has not been observed.   The true relativity wave equation should thus be such that its solutions split up into two non-combining sets, referring respectively to the charge $- e$ and the charge e.

In the present paper we shall be concerned only with the removal of the first of these two difficulties.   The resulting theory is therefore still only an approxi-mation, but it appears to be good enough to account for all the duplexity phenomena without arbitrary assumptions.

§ 2. *The Hamiltonian for No Field.*

Our problem is to obtain a wave equation of the form (2) which shall be invariant under a Lorentz transformation and shall be equivalent to (1) in the limit of large quantum numbers. We shall consider first the case of no field, when equation (1) reduces to

$$(- p_0{}^2 + \mathbf{p}^2 + m^2 c^2)\, \psi = 0 \qquad (3)$$

if one puts

$$p_0 = \frac{\mathrm{W}}{c} = ih\, \frac{\partial}{c\, \partial t}.$$

The symmetry between p_0 and p_1, p_2, p_3 required by relativity shows that, since the Hamiltonian we want is linear in p_0, it must also be linear in p_1, p_2 and p_3. Our wave equation is therefore of the form

$$(p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta)\, \psi = 0, \qquad (4)$$

where for the present all that is known about the dynamical variables or operators α_1, α_2, α_3, β is that they are independent of p_0, p_1, p_2, p_3, *i.e.*, that they commute with t, x_1, x_2, x_3. Since we are considering the case of a particle moving in empty space, so that all points in space are equivalent, we should expect the Hamiltonian not to involve t, x_1, x_2, x_3. This means that α_1, α_2, α_3, β are independent of t, x_1, x_2, x_3, *i.e.*, that they commute with p_0, p_1, p_2, p_3. We are therefore obliged to have other dynamical variables besides the co-ordinates and momenta of the electron, in order that α_1, α_2, α_3, β may be functions of them. The wave function ψ must then involve more variables than merely x_1, x_2, x_3, t.

Equation (4) leads to

$$0 = (- p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta)(p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta)\, \psi$$
$$= [- p_0{}^2 + \Sigma\, \alpha_1{}^2 p_1{}^2 + \Sigma\, (\alpha_1 \alpha_2 + \alpha_2 \alpha_1)\, p_1 p_2 + \beta^2 + \Sigma\, (\alpha_1 \beta + \beta \alpha_1)\, p_1]\, \psi, \quad (5)$$

where the Σ refers to cyclic permutation of the suffixes 1, 2, 3. This agrees with (3) if

$$\left.\begin{array}{ll} \alpha_r{}^2 = 1, & \alpha_r \alpha_s + \alpha_s \alpha_r = 0 \quad (r \neq s) \\[2mm] \beta^2 = m^2 c^2, & \alpha_r \beta + \beta \alpha_r = 0 \end{array}\right\} \quad r, s = 1, 2, 3.$$

If we put $\beta = \alpha_4 mc$, these conditions become

$$\alpha_\mu{}^2 = 1 \qquad \alpha_\mu \alpha_\nu + \alpha_\nu \alpha_\mu = 0\ (\mu \neq \nu) \qquad \mu, \nu = 1, 2, 3, 4. \qquad (6)$$

We can suppose the α_μ's to be expressed as matrices in some matrix scheme, the matrix elements of α_μ being, say, $\alpha_\mu\, (\zeta'\ \zeta'')$. The wave function ψ must

now be a function of ζ as well as x_1, x_2, x_3, t. The result of α_μ multiplied into ψ will be a function $(\alpha_\mu \psi)$ of x_1, x_2, x_3, t, ζ defined by

$$(\alpha_\mu \psi)\,(x,\,t,\,\zeta) = \Sigma_{\zeta'}\,\alpha_\mu\,(\zeta\,\zeta')\,\psi\,(x,\,t,\,\zeta').$$

We must now find four matrices α_μ to satisfy the conditions (6). We make use of the matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which Pauli introduced* to describe the three components of spin angular momentum. These matrices have just the properties

$$\sigma_r{}^2 = 1 \qquad \sigma_r\sigma_s + \sigma_s\sigma_r = 0, \qquad (r \neq s), \tag{7}$$

that we require for our α's. We cannot, however, just take the σ's to be three of our α's, because then it would not be possible to find the fourth. We must extend the σ's in a diagonal manner to bring in two more rows and columns, so that we can introduce three more matrices ρ_1, ρ_2, ρ_3 of the same form as σ_1, σ_2, σ_3, but referring to different rows and columns, thus :—

$$\sigma_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$\rho_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \rho_2 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix} \quad \rho_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

The ρ's are obtained from the σ's by interchanging the second and third rows, and the second and third columns. We now have, in addition to equations (7)

and also

$$\rho_r{}^2 = 1 \qquad \rho_r\rho_s + \rho_s\rho_r = 0 \qquad (r \neq s), \left.\begin{array}{c} \\ \\ \rho_r\sigma_t = \sigma_t\rho_r. \end{array}\right\} \tag{7'}$$

* Pauli, *loc. cit.*

If we now take

$$\alpha_1 = \rho_1\sigma_1, \qquad \alpha_2 = \rho_1\sigma_2, \qquad \alpha_3 = \rho_1\sigma_3, \qquad \alpha_4 = \rho_3,$$

all the conditions (6) are satisfied, *e.g.*,

$$\alpha_1{}^2 = \rho_1\sigma_1\rho_1\sigma_1 = \rho_1{}^2\sigma_1{}^2 = 1$$
$$\alpha_1\alpha_2 = \rho_1\sigma_1\rho_1\sigma_2 = \rho_1{}^2\sigma_1\sigma_2 = -\rho_1{}^2\sigma_2\sigma_1 = -\alpha_2\alpha_1.$$

The following equations are to be noted for later reference

$$\left.\begin{array}{l} \rho_1\rho_2 = i\rho_3 = -\rho_2\rho_1 \\ \sigma_1\sigma_2 = i\sigma_3 = -\sigma_2\sigma_1 \end{array}\right\}, \tag{8}$$

together with the equations obtained by cyclic permutation of the suffixes.

The wave equation (4) now takes the form

$$[p_0 + \rho_1\,(\boldsymbol{\sigma},\,\mathbf{p}) + \rho_3 mc]\,\psi = 0, \tag{9}$$

where $\boldsymbol{\sigma}$ denotes the vector $(\sigma_1,\,\sigma_2,\,\sigma_3)$.

§ 3. *Proof of Invariance under a Lorentz Transformation.*

Multiply equation (9) by ρ_3 on the left-hand side. It becomes, with the help of (8),

$$[\rho_3 p_0 + i\rho_2\,(\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3) + mc]\,\psi = 0.$$

Putting

$$p_0 = ip_4,$$

$$\rho_3 = \gamma_4, \qquad \rho_2\sigma_r = \gamma_r, \qquad r = 1,\,2,\,3, \tag{10}$$

we have

$$[i\Sigma\gamma_\mu p_\mu + mc]\,\psi = 0, \qquad \mu = 1,\,2,\,3,\,4. \tag{11}$$

The p_μ transform under a Lorentz transformation according to the law

$$p_\mu{}' = \Sigma_\nu a_{\mu\nu} p_\nu,$$

where the coefficients $a_{\mu\nu}$ are c-numbers satisfying

$$\Sigma_\mu a_{\mu\nu} a_{\mu\tau} = \delta_{\nu\tau}, \qquad \Sigma_\tau a_{\mu\tau} a_{\nu\tau} = \delta_{\mu\nu}.$$

The wave equation therefore transforms into

$$[i\Sigma\gamma_\mu{}' p_\mu{}' + mc]\,\psi = 0, \tag{12}$$

where

$$\gamma_\mu{}' = \Sigma_\nu a_{\mu\nu} \gamma_\nu.$$

Now the γ_μ, like the α_μ, satisfy

$$\gamma_\mu{}^2 = 1, \qquad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 0, \qquad (\mu \neq \nu).$$

These relations can be summed up in the single equation

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}.$$

We have

$$\gamma_\mu' \gamma_\nu' + \gamma_\nu' \gamma_\mu' = \Sigma_{\tau\lambda} a_{\mu\tau} a_{\nu\lambda} (\gamma_\tau \gamma_\lambda + \gamma_\lambda \gamma_\tau)$$

$$= 2\Sigma_{\tau\lambda} a_{\mu\tau} a_{\nu\lambda} \delta_{\tau\lambda}$$

$$= 2\Sigma_\tau a_{\mu\tau} a_{\nu\tau} = 2\delta_{\mu\nu}.$$

Thus the γ_μ' satisfy the same relations as the γ_μ. Thus we can put, analogously to (10)

$$\gamma_4' = \rho_3' \qquad \gamma_r' = \rho_2' \sigma_r'$$

where the ρ''s and σ''s are easily verified to satisfy the relations corresponding to (7), (7') and (8), if ρ_2' and ρ_1' are defined by $\rho_2' = -i\gamma_1'\gamma_2'\gamma_3'$, $\rho_1' = -i\rho_2'\rho_3'$.

We shall now show that, by a canonical transformation, the ρ''s and σ''s may be brought into the form of the ρ's and σ's. From the equation $\rho_3'^2 = 1$, it follows that the only possible characteristic values for ρ_3' are ± 1. If one applies to ρ_3' a canonical transformation with the transformation function ρ_1', the result is

$$\rho_1'\rho_3' (\rho_1')^{-1} = -\rho_3'\rho_1' (\rho_1')^{-1} = -\rho_3'.$$

Since characteristic values are not changed by a canonical transformation, ρ_3' must have the same characteristic values as $-\rho_3'$. Hence the characteristic values of ρ_3' are $+1$ twice and -1 twice. The same argument applies to each of the other ρ''s, and to each of the σ''s.

Since ρ_3' and σ_3' commute, they can be brought simultaneously to the diagonal form by a canonical transformation. They will then have for their diagonal elements each $+1$ twice and -1 twice. Thus, by suitably rearranging the rows and columns, they can be brought into the form ρ_3 and σ_3 respectively. (The possibility $\rho_3' = \pm \sigma_3'$ is excluded by the existence of matrices that commute with one but not with the other.)

Any matrix containing four rows and columns can be expressed as

$$c + \Sigma_r c_r \sigma_r + \Sigma_r c_r' \rho_r + \Sigma_{rs} c_{rs} \rho_r \sigma_s \tag{13}$$

where the sixteen coefficients c, c_r, c_r', c_{rs} are c-numbers. By expressing σ_1' in this way, we see, from the fact that it commutes with $\rho_3' = \rho_3$ and anticommutes* with $\sigma_3' = \sigma_3$, that it must be of the form

$$\sigma_1' = c_1\sigma_1 + c_2\sigma_2 + c_{31}\rho_3\sigma_1 + c_{32}\rho_3\sigma_2,$$

* We say that a anticommutes with b when $ab = -ba$.

i.e., of the form

$$\sigma_1' = \begin{Bmatrix} 0 & a_{12} & 0 & 0 \\ a_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & a_{43} & 0 \end{Bmatrix}.$$

The condition $\sigma_1'^2 = 1$ shows that $a_{12}a_{21} = 1$, $a_{34}a_{43} = 1$. If we now apply the canonical transformation : first row to be multiplied by $(a_{21}/a_{12})^{\frac{1}{2}}$ and third row to be multiplied by $(a_{43}/a_{34})^{\frac{1}{2}}$, and first and third columns to be divided by the same expressions, σ_1' will be brought into the form of σ_1, and the diagonal matrices σ_3' and ρ_3' will not be changed.

If we now express ρ_1' in the form (13) and use the conditions that it commutes with $\sigma_1' = \sigma_1$ and $\sigma_3' = \sigma_3$ and anticommutes with $\rho_3' = \rho_3$, we see that it must be of the form

$$\rho_1' = c_1'\rho_1 + c_2'\rho_2.$$

The condition $\rho_1'^2 = 1$ shows that $c_1'^2 + c_2'^2 = 1$, or $c_1' = \cos\,\theta$, $c_2' = \sin\,\theta$. Hence ρ_1' is of the form

$$\rho_1' = \begin{Bmatrix} 0 & 0 & e^{-i\theta} & 0 \\ 0 & 0 & 0 & e^{-i\theta} \\ e^{i\theta} & 0 & 0 & 0 \\ 0 & e^{i\theta} & 0 & 0 \end{Bmatrix}.$$

If we now apply the canonical transformation : first and second rows to be multiplied by $e^{i\theta}$ and first and second columns to be divided by the same expression, ρ_1' will be brought into the form ρ_1, and σ_1, σ_3, ρ_3 will not be altered. ρ_2' and σ_2' must now be of the form ρ_2 and σ_2, on account of the relations $i\rho_2' = \rho_3'\rho_1'$, $i\sigma_2' = \sigma_3'\sigma_1'$.

Thus by a succession of canonical transformations, which can be combined to form a single canonical transformation, the ρ''s and σ''s can be brought into the form of the ρ's and σ's. The new wave equation (12) can in this way be brought back into the form of the original wave equation (11) or (9), so that the results that follow from this original wave equation must be independent of the frame of reference used.

§ 4. *The Hamiltonian for an Arbitrary Field.*

To obtain the Hamiltonian for an electron in an electromagnetic field with scalar potential A_0 and vector potential $\mathbf{A}$, we adopt the usual procedure of substituting $p_0 + e/c \,.\, A_0$ for p_0 and $\mathbf{p} + e/c \,.\, \mathbf{A}$ for $\mathbf{p}$ in the Hamiltonian for no field. From equation (9) we thus obtain

$$\left[p_0 + \frac{e}{c} A_0 + \rho_1 \left(\boldsymbol{\sigma}, \, \mathbf{p} + \frac{e}{c} \mathbf{A} \right) + \rho_3 mc \right] \psi = 0. \tag{14}$$

This wave equation appears to be sufficient to account for all the duplexity phenomena. On account of the matrices ρ and σ containing four rows and columns, it will have four times as many solutions as the non-relativity wave equation, and twice as many as the previous relativity wave equation (1). Since half the solutions must be rejected as referring to the charge $+ e$ on the electron, the correct number will be left to account for duplexity phenomena. The proof given in the preceding section of invariance under a Lorentz transformation applies equally well to the more general wave equation (14).

We can obtain a rough idea of how (14) differs from the previous relativity wave equation (1) by multiplying it up analogously to (5). This gives, if we write e' for e/c

$$0 = [- (p_0 + e'A_0) + \rho_1 (\boldsymbol{\sigma}, \, \mathbf{p} + e'\mathbf{A}) + \rho_3 mc]$$
$$\times [(p_0 + e'A_0) + \rho_1 (\boldsymbol{\sigma}, \, \mathbf{p} + e'\mathbf{A}) + \rho_3 mc] \psi$$
$$= [- (p_0 + e'A_0)^2 + (\boldsymbol{\sigma}, \, \mathbf{p} + e'\mathbf{A})^2 + m^2 c^2$$
$$+ \rho_1 \{ (\boldsymbol{\sigma}, \, \mathbf{p} + e'\mathbf{A}) (p_0 + e'A_0) - (p_0 + e'A_0) (\boldsymbol{\sigma}, \, \mathbf{p} + e'\mathbf{A}) \}] \psi. \tag{15}$$

We now use the general formula, that if $\mathbf{B}$ and $\mathbf{C}$ are any two vectors that commute with $\boldsymbol{\sigma}$

$$(\boldsymbol{\sigma}, \, \mathbf{B}) (\boldsymbol{\sigma}, \, \mathbf{C}) = \Sigma \, \sigma_1{}^2 B_1 C_1 + \Sigma \, (\sigma_1 \sigma_2 B_1 C_2 + \sigma_2 \sigma_1 B_2 C_1)$$
$$= (\mathbf{B}, \, \mathbf{C}) + i \, \Sigma \, \sigma_3 (B_1 C_2 - B_2 C_1)$$
$$= (\mathbf{B}, \, \mathbf{C}) + i \, (\boldsymbol{\sigma}, \, \mathbf{B} \times \mathbf{C}). \tag{16}$$

Taking $\mathbf{B} = \mathbf{C} = \mathbf{p} + e'\mathbf{A}$, we find

$$(\boldsymbol{\sigma}, \, \mathbf{p} + e'\mathbf{A})^2 = (\mathbf{p} + e'\mathbf{A})^2 + i \, \Sigma \, \sigma_3$$
$$[(p_1 + e'A_1) (p_2 + e'A_2) - (p_2 + e'A_2)(p_1 + e'A_1)]$$
$$= (\mathbf{p} + e'\mathbf{A})^2 + he' (\boldsymbol{\sigma}, \, \text{curl } \mathbf{A}).$$

Thus (15) becomes

$$0 = \left[-(p_0 + e'\mathbf{A}_0)^2 + (\mathbf{p} + e'\mathbf{A})^2 + m^2 c^2 + e'h\,(\,\boldsymbol{\sigma}\,,\,\text{curl }\mathbf{A})\right.$$

$$\left. - ie'h\rho_1\left(\,\boldsymbol{\sigma}\,,\,\text{grad }\mathbf{A}_0 + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right)\right]\psi$$

$$= \left[-(p_0 + e'\mathbf{A}_0)^2 + (\mathbf{p} + e'\mathbf{A})^2 + m^2 c^2 + e'h\,(\,\boldsymbol{\sigma}\,,\,\mathbf{H}) + ie'h\rho_1\,(\,\boldsymbol{\sigma}\,,\,\mathbf{E})\right]\psi,$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic vectors of the field.

This differs from (1) by the two extra terms

$$\frac{eh}{c}\,(\,\boldsymbol{\sigma}\,,\,\mathbf{H}) + \frac{ieh}{c}\,\rho_1\,(\,\boldsymbol{\sigma}\,,\,\mathbf{E})$$

in F. These two terms, when divided by the factor $2m$, can be regarded as the additional potential energy of the electron due to its new degree of freedom. The electron will therefore behave as though it has a magnetic moment $eh/2mc\,.\,\boldsymbol{\sigma}$ and an electric moment $ieh/2mc\,.\,\rho_1\boldsymbol{\sigma}$. This magnetic moment is just that assumed in the spinning electron model. The electric moment, being a pure imaginary, we should not expect to appear in the model. It is doubtful whether the electric moment has any physical meaning, since the Hamiltonian in (14) that we started from is real, and the imaginary part only appeared when we multiplied it up in an artificial way in order to make it resemble the Hamiltonian of previous theories.

§ 5. *The Angular Momentum Integrals for Motion in a Central Field.*

We shall consider in greater detail the motion of an electron in a central field of force. We put $\mathbf{A} = 0$ and $e'\mathbf{A}_0 = \mathrm{V}\,(r)$, an arbitrary function of the radius r, so that the Hamiltonian in (14) becomes

$$\mathrm{F} \equiv p_0 + \mathrm{V} + \rho_1\,(\,\boldsymbol{\sigma}\,,\,\mathbf{p}) + \rho_3 mc.$$

We shall determine the periodic solutions of the wave equation $\mathrm{F}\,\psi = 0$, which means that p_0 is to be counted as a parameter instead of an operator; it is, in fact, just $1/c$ times the energy level.

We shall first find the angular momentum integrals of the motion. The orbital angular momentum $\mathbf{m}$ is defined by

$$\mathbf{m} = \mathbf{x} \times \mathbf{p},$$

and satisfies the following " Vertauschungs " relations

$$\left.\begin{array}{ll} m_1 x_1 - x_1 m_1 = 0, & m_1 x_2 - x_2 m_1 = ihx_3 \\[4pt] m_1 p_1 - p_1 m_1 = 0, & m_1 p_2 - p_2 m_1 = ihp_3 \\[4pt] \mathbf{m} \times \mathbf{m} = ih\mathbf{m}, & \mathbf{m}^2 m_1 - m_1 \mathbf{m}^2 = 0, \end{array}\right\}, \qquad (17)$$

together with similar relations obtained by permuting the suffixes. Also $\mathbf{m}$ commutes with r, and with p_r, the momentum canonically conjugate to r.

We have

$$m_1 \mathrm{F} - \mathrm{F} m_1 = \rho_1 \{ m_1 (\boldsymbol{\sigma}, \mathbf{p}) - (\boldsymbol{\sigma}, \mathbf{p}) m_1 \}$$

$$= \rho_1 (\boldsymbol{\sigma}, m_1 \mathbf{p} - \mathbf{p} m_1)$$

$$= ih\rho_1 (\sigma_2 p_3 - \sigma_3 p_2),$$

and so

$$\mathbf{m}\mathrm{F} - \mathrm{F}\mathbf{m} = ih\rho_1 \, \boldsymbol{\sigma} \times \mathbf{p}. \tag{18}$$

Thus $\mathbf{m}$ is not a constant of the motion. We have further

$$\sigma_1 \mathrm{F} - \mathrm{F}\sigma_1 = \rho_1 \{ \sigma_1 (\boldsymbol{\sigma}, \mathbf{p}) - (\boldsymbol{\sigma}, \mathbf{p}) \sigma_1 \}$$

$$= \rho_1 (\sigma_1 \boldsymbol{\sigma} - \boldsymbol{\sigma} \sigma_1, \mathbf{p})$$

$$= 2i\rho_1 (\sigma_3 p_2 - \sigma_2 p_3),$$

with the help of (8), and so

$$\boldsymbol{\sigma}\mathrm{F} - \mathrm{F}\boldsymbol{\sigma} = -2i\rho_1 \, \boldsymbol{\sigma} \times \mathbf{p}.$$

Hence

$$(\mathbf{m} + \tfrac{1}{2}h \, \boldsymbol{\sigma}) \mathrm{F} - \mathrm{F} (\mathbf{m} + \tfrac{1}{2}h \, \boldsymbol{\sigma}) = 0.$$

Thus $\mathbf{m} + \tfrac{1}{2}h \, \boldsymbol{\sigma} (= \mathbf{M}$ say) is a constant of the motion. We can interpret this result by saying that the electron has a spin angular momentum of $\tfrac{1}{2}h \, \boldsymbol{\sigma}$, which, added to the orbital angular momentum $\mathbf{m}$, gives the total angular momentum $\mathbf{M}$, which is a constant of the motion.

The Vertauschungs relations (17) all hold when M's are written for the m's. In particular

$$\mathbf{M} \times \mathbf{M} = ih\mathbf{M} \quad \text{and} \quad \mathbf{M}^2 \mathbf{M}_3 = \mathbf{M}_3 \mathbf{M}^2.$$

$\mathbf{M}_3$ will be an action variable of the system. Since the characteristic values of m_3 must be integral multiples of h in order that the wave function may be single-valued, the characteristic values of $\mathbf{M}_3$ must be half odd integral multiples of h. If we put

$$\mathbf{M}^2 = (j^2 - \tfrac{1}{4}) h^2, \tag{19}$$

j will be another quantum number, and the characteristic values of $\mathbf{M}_3$ will extend from $(j - \tfrac{1}{2}) h$ to $(-j + \tfrac{1}{2}) h$.* Thus j takes integral values.

One easily verifies from (18) that $\mathbf{m}^2$ does not commute with F, and is thus not a constant of the motion. This makes a difference between the present theory and the previous spinning electron theory, in which $\mathbf{m}^2$ is constant, and defines the azimuthal quantum number k by a relation similar to (19). We shall find that our j plays the same part as the k of the previous theory.

* See ' Roy. Soc. Proc.,' A, vol. 111, p. 281 (1926).

§ 6. *The Energy Levels for Motion in a Central Field.*

We shall now obtain the wave equation as a differential equation in r, with the variables that specify the orientation of the whole system removed. We can do this by the use only of elementary non-commutative algebra in the following way.

In formula (16) take $\mathbf{B} = \mathbf{C} = \mathbf{m}$. This gives

$$(\boldsymbol{\sigma}, \mathbf{m})^2 = \mathbf{m}^2 + i(\boldsymbol{\sigma}, \mathbf{m} \times \mathbf{m}) \tag{20}$$
$$= (\mathbf{m} + \tfrac{1}{2} h \boldsymbol{\sigma})^2 - h(\boldsymbol{\sigma}, \mathbf{m}) - \tfrac{1}{4} h^2 \boldsymbol{\sigma}^2 - h(\boldsymbol{\sigma}, \mathbf{m})$$
$$= \mathbf{M}^2 - 2h(\boldsymbol{\sigma}, \mathbf{m}) - \tfrac{3}{4} h^2.$$

Hence
$$\{(\boldsymbol{\sigma}, \mathbf{m}) + h\}^2 = \mathbf{M}^2 + \tfrac{1}{4} h^2 = j^2 h^2.$$

Up to the present we have defined j only through j^2, so that we could now, if we liked, take jh equal to $(\boldsymbol{\sigma}, \mathbf{m}) + h$. This would not be convenient since we want j to be a constant of the motion while $(\boldsymbol{\sigma}, \mathbf{m}) + h$ is not, although its square is. We have, in fact, by another application of (16),

$$(\boldsymbol{\sigma}, \mathbf{m})(\boldsymbol{\sigma}, \mathbf{p}) = i(\boldsymbol{\sigma}, \mathbf{m} \times \mathbf{p})$$

since $(\mathbf{m}, \mathbf{p}) = 0$, and similarly

$$(\boldsymbol{\sigma}, \mathbf{p})(\boldsymbol{\sigma}, \mathbf{m}) = i(\boldsymbol{\sigma}, \mathbf{p} \times \mathbf{m}),$$

so that

$$(\boldsymbol{\sigma}, \mathbf{m})(\boldsymbol{\sigma}, \mathbf{p}) + (\boldsymbol{\sigma}, \mathbf{p})(\boldsymbol{\sigma}, \mathbf{m}) = i\Sigma\sigma_1(m_2 p_3 - m_3 p_2 + p_2 m_3 - p_3 m_2)$$
$$= i\Sigma\sigma_1 . 2ih p_1 = -2h(\boldsymbol{\sigma}, \mathbf{p}),$$

or

$$\{(\boldsymbol{\sigma}, \mathbf{m}) + h\}(\boldsymbol{\sigma}, \mathbf{p}) + (\boldsymbol{\sigma}, \mathbf{p})\{(\boldsymbol{\sigma}, \mathbf{m}) + h\} = 0.$$

Thus $(\boldsymbol{\sigma}, \mathbf{m}) + h$ anticommutes with one of the terms in F, namely, $\rho_1(\boldsymbol{\sigma}, \mathbf{p})$, and commutes with the other three. Hence $\rho_3\{(\boldsymbol{\sigma}, \mathbf{m}) + h\}$ commutes with all four, and is therefore a constant of the motion. But the square of $\rho_3\{(\boldsymbol{\sigma}, \mathbf{m}) + h\}$ must also equal $j^2 h^2$. We therefore take

$$jh = \rho_3\{(\boldsymbol{\sigma}, \mathbf{m}) + h\}. \tag{21}$$

We have, by a further application of (16)

$$(\boldsymbol{\sigma}, \mathbf{x})(\boldsymbol{\sigma}, \mathbf{p}) = (\mathbf{x}, \mathbf{p}) + i(\boldsymbol{\sigma}, \mathbf{m}).$$

Now a permissible definition of p_r is

$$(\mathbf{x}, \mathbf{p}) = r p_r + ih,$$

and from (21)

$$(\boldsymbol{\sigma}, \mathbf{m}) = \rho_3 jh - h.$$

Hence

$$(\boldsymbol{\sigma}, \mathbf{x})(\boldsymbol{\sigma}, \mathbf{p}) = r p_r + i\rho_3 jh. \tag{22}$$

Introduce the quantity ε defined by

$$r\varepsilon = \rho_1 \, (\, \boldsymbol{\sigma}, \, \mathbf{x}). \tag{23}$$

Since r commutes with ρ_1 and with $(\, \boldsymbol{\sigma}, \mathbf{x})$, it must commute with ε.   We thus have

$$r^2\varepsilon^2 = [\rho_1 \, (\, \boldsymbol{\sigma}, \, \mathbf{x})\,]^2 = (\, \boldsymbol{\sigma}, \, \mathbf{x})^2 = \mathbf{x}^2 = r^2$$

or

$$\varepsilon^2 = 1.$$

Since there is symmetry between $\mathbf{x}$ and $\mathbf{p}$ so far as angular momentum is concerned, $\rho_1 \, (\, \boldsymbol{\sigma}, \, \mathbf{x})$, like $\rho_1 \, (\, \boldsymbol{\sigma}, \, \mathbf{p})$, must commute with $\mathbf{M}$ and j.   Hence ε commutes with $\mathbf{M}$ and j.   Further, ε must commute with p_r, since we have

$$(\, \boldsymbol{\sigma}, \, \mathbf{x}) \, (\mathbf{x}, \, \mathbf{p}) - (\mathbf{x}, \, \mathbf{p}) \, (\, \boldsymbol{\sigma}, \, \mathbf{x}) = ih \, (\, \boldsymbol{\sigma}, \, \mathbf{x}),$$

which gives

$$r\varepsilon \, (rp_r + ih) - (rp_r + ih) \, r\varepsilon = ihr\varepsilon,$$

which reduces to

$$\varepsilon p_r - p_r\varepsilon = 0.$$

From (22) and (23) we now have

$$r\varepsilon\rho_1 \, (\, \boldsymbol{\sigma}, \, \mathbf{p}) = rp_r + i\rho_3 jh$$

or

$$\rho_1 \, (\, \boldsymbol{\sigma}, \, \mathbf{p}) = \varepsilon p_r + i\varepsilon\rho_3 jh/r.$$

Thus

$$\mathbf{F} = p_0 + \mathbf{V} + \varepsilon p_r + i\varepsilon\rho_3 jh/r + \rho_3 mc. \tag{24}$$

Equation (23) shows that ε anticommutes with ρ_3. We can therefore by a canonical transformation (involving perhaps the x's and p's as well as the σ's and ρ's) bring ε into the form of the ρ_2 of § 2 without changing ρ_3, and without changing any of the other variables occurring on the right-hand side of (24), since these other variables all commute with ε. $i\varepsilon\rho_3$ will now be of the form $i\rho_2\rho_3 = -\rho_1$, so that the wave equation takes the form

$$\mathbf{F}\psi \equiv [p_0 + \mathbf{V} + \rho_2 p_r - \rho_1 jh/r + \rho_3 mc] \, \psi = 0.$$

If we write this equation out in full, calling the components of ψ referring to the first and third rows (or columns) of the matrices ψ_α and ψ_β respectively, we get

$$(\mathbf{F}\psi)_\alpha \equiv (p_0 + \mathbf{V}) \, \psi_\alpha - h\frac{\partial}{\partial r} \psi_\beta - \frac{jh}{r} \psi_\beta + mc\psi_\alpha = 0,$$

$$(\mathbf{F}\psi)_\beta \equiv (p_0 + \mathbf{V}) \, \psi_\beta + h\frac{\partial}{\partial r} \psi_\alpha - \frac{jh}{r} \psi_\alpha - mc\psi_\beta = 0.$$

The second and fourth components give just a repetition of these two equations. We shall now eliminate ψ_a. If we write $h\mathrm{B}$ for $p_0 + \mathrm{V} + mc$, the first equation becomes

$$\left(\frac{\partial}{\partial r} + \frac{j}{r}\right)\psi_\beta = \mathrm{B}\psi_a,$$

which gives on differentiating

$$\frac{\partial^2}{\partial r^2}\psi_\beta + \frac{j}{r}\frac{\partial}{\partial r}\psi_\beta - \frac{j}{r^2}\psi_\beta = \mathrm{B}\frac{\partial}{\partial r}\psi_a + \frac{\partial \mathrm{B}}{\partial r}\psi_a$$

$$\cdot = \frac{\mathrm{B}}{h}\left[-(p_0 + \mathrm{V} - mc)\psi_\beta + \frac{jh}{r}\psi_a\right] + \frac{1}{h}\frac{\partial \mathrm{V}}{\partial r}\psi_a$$

$$= -\frac{(p_0 + \mathrm{V})^2 - m^2c^2}{h^2}\psi_\beta + \left(\frac{j}{r} + \frac{1}{\mathrm{B}h}\frac{\partial \mathrm{V}}{\partial r}\right)\left(\frac{\partial}{\partial r} + \frac{j}{r}\right)\psi_\beta.$$

This reduces to

$$\frac{\partial^2}{\partial r^2}\psi_\beta + \left[\frac{(p_0 + \mathrm{V})^2 - m^2c^2}{h^2} - \frac{j(j+1)}{r^2}\right]\psi_\beta - \frac{1}{\mathrm{B}h}\frac{\partial \mathrm{V}}{\partial r}\left(\frac{\partial}{\partial r} + \frac{j}{r}\right)\psi_\beta = 0. \quad (25)$$

The values of the parameter p_0 for which this equation has a solution finite at $r = 0$ and $r = \infty$ are $1/c$ times the energy levels of the system. To compare this equation with those of previous theories, we put $\psi_\beta = r\chi$, so that

$$\frac{\partial^2}{\partial r^2}\chi + \frac{2}{r}\frac{\partial}{\partial r}\chi + \left[\frac{(p_0 + \mathrm{V})^2 - m^2c^2}{h^2} - \frac{j(j+1)}{r^2}\right]\chi - \frac{1}{\mathrm{B}h}\frac{\partial \mathrm{V}}{\partial r}\left(\frac{\partial}{\partial r} + \frac{j+1}{r}\right)\chi = 0.$$

$$(26)$$

If one neglects the last term, which is small on account of B being large, this equation becomes the same as the ordinary Schroedinger equation for the system, with relativity correction included. Since j has, from its definition, both positive and negative integral characteristic values, our equation will give twice as many energy levels when the last term is not neglected.

We shall now compare the last term of (26), which is of the same order of magnitude as the relativity correction, with the spin correction given by Darwin and Pauli. To do this we must eliminate the $\partial\chi/\partial r$ term by a further transformation of the wave function. We put

$$\chi = \mathrm{B}^{-\frac{1}{2}}\chi_1,$$

which gives

$$\frac{\partial^2}{\partial r^2}\chi_1 + \frac{2}{r}\frac{\partial}{\partial r}\chi_1 + \left[\frac{(p_0 + \mathrm{V})^2 - m^2c^2}{h^2} - \frac{j(j+1)}{r^2}\right]\chi_1$$

$$+ \left[\frac{1}{\mathrm{B}h}\frac{j}{r}\frac{\partial \mathrm{V}}{\partial r} - \frac{1}{2}\frac{1}{\mathrm{B}h}\frac{\partial^2 \mathrm{V}}{\partial r^2} + \frac{1}{4}\frac{1}{\mathrm{B}^2h^2}\left(\frac{\partial \mathrm{V}}{\partial r}\right)^2\right]\chi_1 = 0. \quad (27)$$

2 υ 2

The correction is now, to the first order of accuracy

$$\frac{1}{\mathrm{B}h}\left(\frac{j}{r}\frac{\partial \mathrm{V}}{\partial r} - \tfrac{1}{2}\frac{\partial^2 \mathrm{V}}{\partial r^2}\right),$$

where $\mathrm{B}h = 2mc$ (provided p_0 is positive). For the hydrogen atom we must put $\mathrm{V} = e^2/cr$. The first order correction now becomes

$$-\frac{e^2}{2mc^2 r^3}(j+1). \tag{28}$$

If we write $-j$ for $j+1$ in (27), we do not alter the terms representing the unperturbed system, so

$$\frac{e^2}{2mc^2 r^3}\, j \tag{28'}$$

will give a second possible correction for the same unperturbed term.

In the theory of Pauli and Darwin, the corresponding correcting term is

$$\frac{e^2}{2mhc^2 r^3}(\boldsymbol{\sigma}, \mathbf{m})$$

when the Thomas factor $\tfrac{1}{2}$ is included. We must remember that in the Pauli-Darwin theory, the resultant orbital angular momentum k plays the part of our j. We must define k by

$$\mathbf{m}^2 = k(k+1)h^2$$

instead of by the exact analogue of (19), in order that it may have integral characteristic values, like j. We have from (20)

$$(\boldsymbol{\sigma}, \mathbf{m})^2 = k(k+1)h^2 - h(\boldsymbol{\sigma}, \mathbf{m})$$

or

$$\{(\boldsymbol{\sigma}, \mathbf{m}) + \tfrac{1}{2}h\}^2 = (k+\tfrac{1}{2})^2 h^2,$$

hence

$$(\boldsymbol{\sigma}, \mathbf{m}) = kh \text{ or } -(k+1)h.$$

The correction thus becomes

$$\frac{e^2}{2mc^2 r^3}\, k \quad \text{or} \quad -\frac{e^2}{2mc^2 r^3}(k+1),$$

which agrees with (28) and (28'). The present theory will thus, in the first approximation, lead to the same energy levels as those obtained by Darwin, which are in agreement with experiment.